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STUDY OF INTEGRAL EQUATIONS USED IN VARIOUS FIELDS OF SCIENCE AND ENGINEERING

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ABSTRACT:

In several branches of science and engineering, integral equations are among the most often used techniques. Mathematical physics, mechanics, the physical sciences, engineering, etc. all rely on integral equations to help them to solve physical problems.

In this study, we will go through some basic definitions and classifications of integral equations. We will present some integral equations in many physical issues arises in various fields and their applications. Further we will discuss how to resolve linear & non-linear integral equations with various techniques viz. Numerical methods, Analytical methods and Laplace transform methods etc.

Keywords :- Integral Equations, Fredholm Integral Equations, Volterra integral equations, Linear Integral Equations, Numerical Examples, Nonlinear Integral Equations.

INTRODUCTION :

One of the most effective tools in mathematics' both in practical and theoretical situations, is the integral equation. They are really useful for solving many physical issues like initial value and boundary value issues with ODE and PDE may be transformed into issues with approximating integral equations. The theory of Fourier's integral is where integral equations were originally discovered. Abel developed a further integral equation in 1826. The work of the Swedish mathematician I. Fredholm (1900) and the Italian mathematician V. Volterra in 1896 marked the beginning of the true development of the integral equations theory.

The present study is mainly concerned with linear integral equations, in particular with Fredholm and Volterra-type of integral equations. One of the main relevancies of this study arises from the idea of transforming differential equations into integral equations and vice versa. Additionally, it contains a few numerical techniques for resolving these equations. Also, the study presents some applications of linear integral equations.

Integral equation:

Any equation that has the unknown functions with integral sign is known as having an integral equation. The typical integral equations have the following form as under:

"$g(x) = f(x) + \lambda \int_{\alpha((x)}^{\beta(x)} k(x,t)\, g(t)dt$" .... (1)

Now, this integral equation kernel is represented by $k(x,t)$, the integration limits by $\alpha(x)$, $\beta(x)$, a constant parameter by λ. It is simple to see that the unknown g(x) function has an integral sign. It should be noticed that the functions f (x), kernel $k(x,t)$ in equation (1) are provided functions. Our primary goal is to use a range of

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solution methods to identify the unknown function g(x) that may meet equation (1).

Classification of Integral Equations

There are several categories that an integral equation can come into. We can classify integral equations as Homogeneous or nonhomogeneous, non-linear or linear. The two primary kinds of integral equations that are most often utilized are the Fredholm and Volterra integral equations. An integral equation is considered to be linear if the unknown function is only subjected to linear operations. The functional form F(g(x)) of the unknown g(x) function with sign of the integral, is presented in a nonlinear integral equation such that the power of g(x) is not unity now.

The form of the most common kind of linear integral equation is

$\phi(x)g(x) = f(x) + \lambda \int a\ k\ (x,\ t)\ g(t)dt$ .....(2)

where this upper value is either a fixed or a variable limit. These functions $\phi(x)$, f (x), $k\ (x,\ t)$ are well-known, however g(x) is unknown and λ is the nonzero real or complex parameter.

Equation (2) is referred to be homogeneous if f (x) = 0, else it is non-homogeneous. From this entire study mainly, we focus on linear integral equations.

Fredholm integral equations

These equations with a defined upper limit b are often expressed by

"$\phi(x)g(x) = f(x) + \lambda \int_a^b K(x,t)g(t)dt$ .....(3)

It is of 1st kind, if $\phi(x)$=0 then (3) becomes $f(x) + \lambda \int_a^b K(x,t)g(t)dt = 0$ .....(4)

and of 2nd kind, if $\phi(x)$ =1 then (3) becomes $g(x) = (x) + \lambda \int_a^b K(x,t)g(t)dt$" .....(5)

Volterra integral equations: These typical Volterra linear integral equations type with upper limit variable x rather than the constant is represented by

"$\phi(x)\ g(x) = f(x) + \lambda \int_a^x K(x,t)g(t)dt$ .....(6)

It is of 1st type, If $\phi(x)$ =0 then (6) becomes $(x) + \lambda \int_a^x K(x,t)g(t)dt = 0$ .....(7)

and of 2nd kind, If $\phi(x)$ =1 then (6) becomes $g\ (x) = (x) + \lambda \int_a^x K(x,t)g(t)dt$" .....(8)

Integro-differential equations is integral equations in which unknown u(x) function occurs in addition to one or more of its derivatives, like $g'(x), g''(x), \ldots$ with integral sign and as an integration of the ordinary derivative and with integral sign.

Volterra-Fredholm integral equations

This equation combines the two distinct integrals, Volterra and Fredholm. These equations are derived from a range of chemical and physical applications, boundary value issues, and models of the spatiotemporal evolution of epidemic. The standard form of this equation looks like:

$g\ (x) = f(x) + \int_0^x K_1(x,t)g(t)dt + \int_a^b K_2(x,t)g(t)dt$ **.....**(9)

where kernels of the equation are $K_1(x,t)\ \&\ K_2(x,t)$.

Volterra-Fredholm Integro-Differential Equations

One integral equation may include Volterra-Fredholm Differential Equation, which integrates the disjoint integrals of Volterra and Fredholm with the Differential operator. These equations are derived from several physical and chemical applications. These applications result in Volterra-Fredholm equations. The standard equation form looks like:

$g^n\ (x) = f(x) + \int_0^x K_1(x,t)g(t)dt + \int_a^b K_2(x,t)g(t)dt$ .....(10)

where the kernels of the equation are $K_1(x,t)\ \&\ K_2(x,t)$. Additionally, the ordinary derivative of g(x) is of order n. Be cautious to note that depending on the included derivative's order, initial conditions must be specified since this type of eqs. comprises ordinary derivatives.

Various Methods to solve Integral Equations

The two significant conventional approaches are methods of successive approximations as well as the successive substitution method. Additionally, the direct computational as well as

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series solution methods are appropriate for the same issues. During last two decades, some methods are developed such as ADM (Adomian decomposition method), the modified decomposition as well as Picard's method. These methods are very popular amongst the engineers and scientist for solving highly nonlinear integral equations. Abel can easily solve the singular integral equation with Laplace transform approach. This method can be utilized for solving the Volterra integral problem of the convolution type. Adomian developed numerical method also known as the ADM or simply decomposition method which gives the solution in series form. Here, we provide a few strategies for solving linear integral equations.

Conversion method of IVP related to a Volterra integral equation

Many physical problems arising in the field of physics and engineering are mathematically represented by the differential equation with initial conditions and the Volterra integral equation may be used to solve these problems.

Consider the differential equation of nth order is of the form is given by

$$\frac{d^n g}{dx^n}+a_1\frac{d^{n-1}g}{dx^{n-1}}+a_2\frac{d^{n-2}g}{dx^{n-2}}+\cdots+a_n u=F(x) \quad .....(11)$$

with $a_1, a_2, \ldots a_n$ are continuous coefficients, together with initial condition

$$g(a)=c_0, g'(a)=c_1, g''(a)=c_{2,} \ldots g^{n-1}(a)=c_{n-1}$$

is referred to as an IVP (initial value problem), for example, a differential equation having an initial value. Here, IVP may be converted into an equivalent Volterra integral equation of 2nd type as:

$$u(x) = f(x) + \int_0^x K(x,t)u(t)dt \quad .....(12)$$

assuming unknown function such that $\frac{d^n g}{dx^n} = u(x)$.

Adomian decomposition method

To demonstrate the ADM viability, several research papers have been published in the domains of chemistry, biology, physics, and mathematics. These publications resolve linear and non-linear differential and integral equations. The estimated solution of a non-linear problem is seen by the ADM as an infinite series that eventually converges to the precise value. ADM is recommended in the study to resolve some 1st, 2nd, and 3rd order differential and integral equations. George Adomian originally presented the ADM in 1981. Early in 1990, Adomian published the approach in his books [1-2] as well as other relevant publications [3-4]. The approach is essentially a power series technique, comparable to the perturbation method.

Proposed method to solve Volterra integral equation:

Consider this equation as u(x) = f(x)

$$+\lambda\int_0^x K(x,t)u(t)dt \quad .....(13)$$

where λ is a parameter and kernel as well as functions are predetermined. The ADM is used in this section. The ADM entails adding an infinite series of components provided by decomposition sequence to the unknown u(x) function of equation "$u(x) = \sum_{n=0}^{\infty} u_n(x)$ .....(14)

Or correspondingly $u(x) = u_0(x) + u_2(x) + u_3(x) + \cdots$

Using these components, $u_n(x)$, $n \geq 0$, they are resolved. Substitute equation (14) into equation (13)

$$\sum_n^\infty u_n(x) = f(x) + \lambda\int_0^x K(x,t)[\textstyle\sum_n^\infty u_n(t)]dt\text{"} \quad .....(15)$$

All terms that do not include zeroth components with integral sign may be used to identify zeroth component, $u_0(x)$. This shows that recurrence relation fully resolves the unknown function components $u(x), u_n(x)$, $n\geq 0$ as

$u_0(x) = f(x)$ and $u_{n+1}(x) = \lambda\int_0^x K(x,t)u_n(t)dt$, $n \geq 0$

i.e. $u_0(x) = f(x)$

$u_1(x) = \lambda\int_0^x K(x,t)u_0(t)dt$

$u_2(x) = \lambda\int_0^x K(x,t)u_1(t)dt$

………………………………

………………………………

$u_n(x) = \lambda\int_0^x K(x,t)u_{n-1}(t)dt.$

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Thus the constituents $u_0(x)$, $u_1(x)$, $u_2(x)$),.... are fully solved. Therefore, with the series as an assumption in the Volterra integral equation (13) allows for the easy attainment of the solution in form of series (14).

The method of Laplace transforms

In many engineering issues, particularly in electrical engineering circuits, the Laplace transform method is crucial.

Consider convolution-type Volterra-integral equation, like

"$u(x) = f(x) + \lambda \int_0^x K(x-t)u(t)dt$" .....(16)

If Laplace transform technique may be used to address the problem relatively quickly and kernels $K(x-t)$ is of the convolution type [1].

Using definition of Laplace transform & convolution integral,

$L\{u(x)\} = \int_0^\infty e^{-sx}u(x)dx$ ..... (17)

"$L\{\int_0^x K(x-t)u(t)dt\} = L\{K(x)\}L\{u(x)\}$ .....(18)

If Laplace transform is applied to equation (16), it can be deduced into

$L\{u(x)\} = L\{f(x)\} + \lambda L\{K(x)\}L\{u(x)\}$ ..... (19)

Now, solution is given by

$L\{u(x)\} = \frac{L\{f(x)\}}{1-\lambda L\{K(x)\}}$ ..... (20)

Thus, $u(x) = \int_0^x \psi(x-t)f(t)dt$ .....(21)

where $\psi = L^{-1}\left\{\frac{1}{1-\lambda L\{K(x)\}}\right\}$"

is the solution of 2nd type Volterra integral equation.

Applications:

Example 1. Consider the IVP equation

$u''(t) + 4u'(t) = \sin t$; at $t = 0, u(0) = 0\ u'(0) = 0$ ..... (22)

assuming the transformation

$u''(t) = g(t)$,

Integrating this equation w.r.t 't' from 0 to t, we have

$u'(t) = \int_0^t g(x)dx$,

again, integrating we obtain

$u(t) = \int_0^t \int_0^t g(x)dx\,dx$,

which implies that

$u(t) = \int_0^t (t-x)g(x)dx$,

put all above values in equation (22), we get

$g(t) = 4\int_0^t \int_0^t (t-x)g(x)\,dx = \sin t$, It represents the second type of Volterra integral equation.

Example 2: We consider the equations

$u(t) = t + \int_0^t (x-t)u(x)dx$ ..... (23)

Let $u(t) = \sum_{n=0}^{\infty} u_n(t)$

Equation (23) then becomes by using ADM

"$\sum_{n=0}^{\infty} u_n(t) = t + \int_0^t (x-t)(\sum_{n=0}^{\infty} u_n(x))dx$ ..... (24)

Decomposing the different terms, we get the set of solution

$u_0(t) = t$

$u_1(t) = \int_0^t (x-t)u_0(x)dx = \int_0^t (x-t)x dx = \frac{-t^3}{3!}$"

$u_2(t) = \int_0^t (x-t)u_1(x)dx = \int_0^t (x-t)\frac{-x^3}{3!}dx = \frac{-t^5}{5!}$

and so on.

Now from equation (24) the solution will be,

$u(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots = \sin t.$

Thus $u(t) = \sin t$ is the solution for Volterra equation (23)

Example 3: Let Fredholm integral equation, for instance.

$u(t) = e^t - 1 + \int_0^1 x\,u(x)dx$ ..... (25).

Let $u(t) = \sum_{n=0}^{\infty} u_n(t)$

Equation (25), as a consequence of the ADM, becomes:

$\sum_{n=0}^{\infty} u_n(t) = e^t - 1 + \int_0^1 x(\sum_{n=0}^{\infty} u_n(x))dx$ ..... (26)

Decomposing the different terms, we get the set of solution

$u_0(t) = e^t - 1$

$u_1(t) = \int_0^1 xu_0(x)dx = \int_0^1 x(e^x - 1)dx = \frac{1}{2}$,

$u_2(t) = \int_0^1 xu_1(x)dx = \int_0^1 \frac{x}{2}dx = \frac{1}{4}$,

$u_3(t) = \int_0^1 xu_2(x)dx = \int_0^1 \frac{x}{4}dx = \frac{1}{8}$,

....

....

and so on.

Now from equation (24) the solution will be,

$$u(t) = e^t - 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = e^t$$

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CONCLUSION:

In this present paper we focus on the results of solving the Volterra and Fredholm integral equations. Here, we used an integral equation decomposition approach to solve an initial value issue involving an ordinary differential equation. Further we discussed some applications through examples. As we solved the numerical instances, it conclude that Volterra and Fredholm integral equations of second type can be solved using ADM with reduced error when there is growing 'n' order.

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